A Mathematical Proof of Taylor Exponent

Here we show that the Taylor exponent of an independent and identically distributed (i.i.d.) process is 0.5. A proof in a more general form is shown in (Eisler, Bartos, and Kertész, 2007). This is a known mathematical fact, as found previously in (Yule, 1968).

**Proposition 1.** The Taylor exponent of a sequence generated by an i.i.d. process is 0.5.

**Proof.** Consider i.i.d. random variables $X_1, \ldots, X_i, \ldots, X_N$, where $i$ denotes the location within a text. For a specific word $w_k \in W$, with $W$ being the set of words, let $p_k$ denote the probability of occurrence of word $w_k$, i.e., $P(X_i = w_k) = p_k$ (for all $i$). Naturally, the expectation $E$ and variance $\mathbb{V}$ of the count of $w_k$ for $X_i$ are the following:

$$E[X_i] = p_k, \quad (1)$$
$$\mathbb{V}[X_i] = p_k(1 - p_k), \quad (2)$$

which only depend on the constant $p_k$. With window size $\Delta t$, $\mu_k = \Delta t E[X_i]$. Note that $\sigma_k^2 = \Delta t \mathbb{V}[X_i]$, because

$$\sigma_k^2 = \mathbb{V} \left[ \sum_{i=1}^{\Delta t} X_i \right] = E \left[ \left( \sum_{i=1}^{\Delta t} (X_i - p_k) \right)^2 \right] = E \left[ \sum_{i=1}^{\Delta t} (X_i - p_k)^2 \right] + \sum_{i \neq j}^{\Delta t} (X_i - p_k)(X_j - p_k)$$

Furthermore, note that $E[(X_i - p_k)(X_j - p_k)] = 0$ for every $i, j$ with $i \neq j$, because $X_i$ and $X_j$ are independent of each other and (1) holds. Therefore, Taylor exponent $\alpha$ of an i.i.d. process is 0.5, because

$$\sigma_k^2 = \mathbb{V}[X_i] \mu_k.$$

$\square$

**References**
