On the Practical Computational Power of Finite Precision RNNs for Language Recognition

Gail Weiss, Yoav Goldberg, Eran Yahav

GRU < LSTM (!?)
Current State

• RNNs are everywhere

• We don’t know too much about the differences between them:
  • Gated RNNs are shown to train better, beyond that:
    • “RNNs are Turing Complete”? 
Turing Complete?

On the Computational Power of Neural Nets*

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Received February 4, 1992; revised May 24, 1993
Turing Complete?

1993 Proof:

1. Requires Infinite Precision:
   Uses stack(s), maintained in certain dimension(s)
   Zeros are pushed using division (using $g = g/4 + 1/4$)
   In 32 bits, this reaches the limit after 15 pushes

2. Requires Infinite Time:
   Allows processing steps beyond reading input
   (Not the standard use case!)

unreasonable assumptions!
Turing Complete?

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   Allows processing steps beyond reading input
   (Not the standard use case!)

unreasonable assumptions!
What happens on real hardware and real use-cases?
Real Use

- Gated architectures have the best performance
  - LSTM and GRU are most popular
  - Of these, the choice between them is unclear
Main Result

We accept all RNN types can simulate DFAs

We show that LSTMs and IRNNs can also count

And that the GRU and SRNN cannot
Power of Counting

Practical

In NMT:
LSTM better at capturing target length
Power of Counting

Practical

In NMT:
LSTM better at capturing target length

Theoretical

Finite State Machines vs Counter Machines
K-Counter Machines (SKCMs)

Fischer, Meyer, Rosenberg - 1968

• Similar to finite automata, but also maintain k counters

• A counter has 4 operations: inc/dec by one, do nothing, reset

• Counters are observed by comparison to zero
Counting Machines and Chomsky Hierarchy

- Regular Languages (RL)
- Context Free Languages (CFL)
- Context Sensitive Languages (CSL)
- Recursively Enumerable Languages (RE)
Chomsky Hierarchy and SKCMs
Chomsky Hierarchy and SKCMs

Regular Languages (RL)

Context Free Languages (CFL)

Context Sensitive Languages (CSL)

Recursively Enumerable Languages (RE)

Palindromes

$\text{a}^n\text{b}^n$

$\text{a}^n\text{b}^n\text{c}^n$
Chomsky Hierarchy and SKCMs

Regular Languages (RL)

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Palindromes

$ab^n$
Chomsky Hierarchy and SKCMs

Regular Languages (RL)

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Palindromes

$\text{Re}^{n}b^{n}$

$\text{Re}^{n}b^{n}c^{n}$
Chomsky Hierarchy and SKCMs

SKCMs cross the Chomsky Hierarchy!

- Regular Languages (RL)
- Context Free Languages (CFL)
- Context Sensitive Languages (CSL)
- Recursively Enumerable Languages (RE)

Palindromes

$anbn$  $anbn^2$  $anbncn$
Summary so Far

- Counters give additional formal power
- We claimed that LSTM can count and GRU cannot
Summary so Far

• Counters give additional formal power

• We claimed that LSTM can count and GRU cannot

• Let’s see why
Popular Architectures

**GRU**

\[
z_t = \sigma(W^z x_t + U^z h_{t-1} + b^z)
\]

\[
r_t = \sigma(W^r x_t + U^r h_{t-1} + b^r)
\]

\[
\tilde{h}_t = \tanh(W^h x_t + U^h (r_t \odot h_{t-1}) + b^h)
\]

\[
h_t = z_t \odot h_{t-1} + (1 - z_t) \odot \tilde{h}_t
\]

**LSTM**

\[
f_t = \sigma(W^f x_t + U^f h_{t-1} + b^f)
\]

\[
i_t = \sigma(W^i x_t + U^i h_{t-1} + b^i)
\]

\[
o_t = \sigma(W^o x_t + U^o h_{t-1} + b^o)
\]

\[
\tilde{c}_t = \tanh(W^c x_t + U^c h_{t-1} + b^c)
\]

\[
c_t = f_t \odot c_{t-1} + i_t \odot \tilde{c}_t
\]

\[
h_t = o_t \odot g(c_t)
\]
Popular Architectures

**GRU**

\[
\begin{align*}
z_t &= \sigma(W^z x_t + U^z h_{t-1} + b^z) \\
r_t &= \sigma(W^r x_t + U^r h_{t-1} + b^r) \\
\tilde{h}_t &= \tanh(W^h x_t + U^h (r_t \circ h_{t-1}) + b^h) \\
h_t &= z_t \circ h_{t-1} + (1 - z_t) \circ \tilde{h}_t
\end{align*}
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f_t &= \sigma(W^f x_t + U^f h_{t-1} + b^f) \\
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c_t &= f_t \circ c_{t-1} + i_t \circ \tilde{c}_t \\
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Popular Architectures

**GRU**

\[ z_t \in (0,1) \]
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\[ \tilde{h}_t = \tanh(W^h x_t + U^h (r_t \circ h_{t-1}) + b^h) \]
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Popular Architectures

GRU

\[ z_t \in (0,1) \]
\[ r_t \in (0,1) \]
\[ \tilde{h}_t \in (-1,1) \]
\[ h_t = z_t \cdot h_{t-1} + (1 - z_t) \cdot \tilde{h}_t \]

LSTM

\[ f_t \in (0,1) \]
\[ i_t \in (0,1) \]
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## Popular Architectures

### GRU

- $z_t \in (0,1)$
- $r_t \in (0,1)$
- $\tilde{h}_t \in (-1,1)$
- $h_t = z_t \cdot h_{t-1} + (1 - z) \cdot \tilde{h}_t$

### LSTM

- $f_t \in (0,1)$
- $i_t \in (0,1)$
- $o_t \in (0,1)$
- $\tilde{c}_t \in (-1,1)$
- $c_t = f_t \cdot c_{t-1} + i_t \cdot \tilde{c}_t$
- $h_t = o_t \cdot g(c_t)$
Popular Architectures

**GRU**

\[
\begin{align*}
z_t & \in (0,1) \\
r_t & \in (0,1) \\
\tilde{h}_t & \in (-1,1) \\
h_t &= z_t \cdot h_{t-1} + (1 - z) \cdot \tilde{h}_t
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Interpolation
Popular Architectures

**GRU**

\[ z_t \in (0, 1) \]
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**LSTM**

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\[ c_t = f_t \cdot c_{t-1} + i_t \cdot \tilde{c}_t \]
\[ h_t = o_t \cdot g(c_t) \]

Interpolation

Bounded!
Popular Architectures

**GRU**

- \( z_t \in (0,1) \)
- \( r_t \in (0,1) \)
- \( \tilde{h}_t \in (-1,1) \)

\[
\begin{align*}
    h_t &= z_t \cdot h_{t-1} + (1 - z) \cdot \tilde{h}_t \\
    \text{Interpolation}
\end{align*}
\]

**LSTM**

- \( f_t \in (0,1) \)
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Popular Architectures

**GRU**

- \( z_t \in (0,1) \)
- \( r_t \in (0,1) \)
- \( \tilde{h}_t \in (-1,1) \)
- \( h_t = z_t \cdot h_{t-1} + (1 - z) \cdot \tilde{h}_t \)

**LSTM**

- \( f_t \approx 1 \)
- \( i_t \approx 1 \)
- \( o_t \in (0,1) \)
- \( \tilde{c}_t \in (-1,1) \)
- \( c_t \approx c_{t-1} + \tilde{c}_t \)
- \( h_t = o_t \cdot g(c_t) \)

Interpolation

Addition

\[ c_t = f_t \cdot c_{t-1} + i_t \cdot \tilde{c}_t \]
Popular Architectures

**GRU**

- $z_t \in (0,1)$
- $r_t \in (-1,1)$
- $\tilde{h}_t \in (-1,1)$
- $h_t = z_t \cdot h_{t-1} + (1 - z) \cdot \tilde{h}_t$

**Bounded!**

**LSTM**

- $f_t \approx 1$
- $i_t \approx 1$
- $o_t \in (0,1)$
- $\tilde{c}_t \approx 1$
- $c_t \approx c_{t-1} + 1$
- $h_t = o_t \circ g(c_t)$

Interpolation

Increase by 1

$$c_t = f_t \cdot c_{t-1} + i_t \cdot \tilde{c}_t$$
Popular Architectures

**GRU**

\[ z_t \in (0,1) \]
\[ r_t \in (0,1) \]
\[ \tilde{h}_t \in (-1,1) \]
\[ h_t = z_t \cdot h_{t-1} + (1-z) \cdot \tilde{h}_t \]

**LSTM**

\[ f_t \approx 1 \]
\[ i_t \approx 1 \]
\[ o_t \in (0,1) \]
\[ \tilde{c}_t \approx -1 \]
\[ c_t \approx c_{t-1} - 1 \]
\[ h_t = o_t \cdot g(c_t) \]

Interpolation

**Decrease by 1**

\[ c_t = f_t \cdot c_{t-1} + i_t \cdot \tilde{c}_t \]
Popular Architectures

**GRU**

- $z_t \in (0,1)$
- $r_t \in (-1,1)$
- $\tilde{h}_t \in (-1,1)$
- $h_t = z_t \cdot h_{t-1} + (1 - z) \cdot \tilde{h}_t$

**LSTM**

- $f_t \approx 1$
- $i_t \approx 0$
- $o_t \in (0,1)$
- $\tilde{c}_t \in (-1,1)$
- $c_t \approx c_{t-1}$
- $h_t = o_t \cdot g(c_t)$

**Interpolation**

**Do Nothing**

- $c_t = f_t \cdot c_{t-1} + i_t \cdot \tilde{c}_t$

Bounded!
## Popular Architectures

### GRU

- \( z_t \in (0,1) \)
- \( r_t \in (0,1) \)
- \( \tilde{h}_t \in (-1,1) \)
- \( h_t = z_t \circ h_{t-1} + (1 - z) \circ \tilde{h}_t \)

### LSTM

- \( f_t \approx 0 \)
- \( i_t \approx 0 \)
- \( o_t \in (0,1) \)
- \( \tilde{c}_t \in (-1,1) \)
- \( c_t \approx 0 \)
- \( h_t = o_t \circ g(c_t) \)

### Interpolation

### Reset

\[ c_t = f_t \circ c_{t-1} + i_t \circ \tilde{c}_t \]
Popular Architectures

<table>
<thead>
<tr>
<th>GRU</th>
<th>LSTM</th>
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<tbody>
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<td>( z_t \in (0,1) )</td>
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<tr>
<td>Interpolation</td>
<td>Reset</td>
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</table>
Other Architectures

SRNN

\[ h_t = \sigma_h(W_h x_t + U_h h_{t-1} + b_h) \]

IRNN

\[ h_t = \max(0, W_h x_t + U_h h_{t-1} + b_h) \]
Other Architectures

SRNN

\[ h_t = \sigma_h(W_h x_t + U_h h_{t-1} + b_h) \in (0,1) \]

IRNN

\[ h_t = \max(0, W_h x_t + U_h h_{t-1} + b_h) \]

Bounded!
Other Architectures

**SRNN**

\[ h_t = \sigma_h(W_h x_t + U_h h_{t-1} + b_h) \in (0,1) \]

**IRNN**

\[ h_t = \max(0, W_h x_t + U_h h_{t-1} + b_h) \]

- **Bounded!**
  - keep/reset
  - +0 / +1
  - (subtraction in parallel, also increasing, counter)
Other Architectures

SRNN

\[ h_t = \sigma_h(W_h x_t + U_h h_{t-1} + b_h) \in (0,1) \]

Bounded!

IRNN

\[ h_t = \max(0, W_h x_t + U_h h_{t-1} + b_h) \]

Can Count!

(subtraction in parallel, also increasing, counter)

keep/reset

+0 / +1
So:

- LSTM can count!
- GRU cannot
- Counting gives greater computational power
Empirically

Trained \( a^n b^n \), (on positive examples up to length 100)

Activations on \( a^{1000} b^{1000} \):

LSTM

GRU
Empirically

Trained $a^n b^n$, (on positive examples up to length 100)

Activations on $a^{1000} b^{1000}$:

LSTM

GRU

GRU:
- Took much longer to train
Empirically

Trained $a^n b^n$, (on positive examples up to length 100)

Activations on $a^{1000} b^{1000}$:

**LSTM**

**GRU**

**GRU:**

- Took much longer to train
- Did not generalise even within training domain
  - begin failing at $n=39$ (vs 257 for LSTM)
Empirically

Trained $a^n b^n$, (on positive examples up to length 100)

Activations on $a^{1000} b^{1000}$:

**LSTM**

**GRU**

**GRU:**
- Took much longer to train
- Did not generalise even within training domain
  - begin failing at $n=39$ (vs 257 for LSTM)
- Did not learn any discernible counting mechanism
Empirically

Trained $a^n b^n c^n$, (on positive examples up to length 50)

Activations on $a^{100} b^{100} c^{100}$:
Empirically

Trained $a^n b^n c^n$, (on positive examples up to length 100)

Activations on $a^{100} b^{100} c^{100}$:

**LSTM**

**GRU**

**GRU:**
- Took much longer to train
- Did not generalise well
  - begin failing at $n=9$ (vs 101 for LSTM)
- Did not learn any discernible counting mechanism
Conclusion

<table>
<thead>
<tr>
<th>IRNN</th>
<th>SRNN</th>
<th>LSTM</th>
<th>GRU</th>
<th>Trainability</th>
</tr>
</thead>
</table>

Conclusion

Practical Expressivity

IRNN     SRNN

LSTM     GRU

Trainability
Take Home Message

Architectural Choices Matter!

and result in actual differences in expressive power

Don’t fall in the Turing Tarpit!
Thank You

GitHub repository:

https://github.com/tech-srl/counting_dimensions

Google Colab (link through GitHub as well):

https://tinyurl.com/ybijkumrz